Math 429 - Exercise Sheet 4

1. Find an example of an element of $SL_2(\mathbb{R})$ which is not in the image of the exponential map.

2. Show that a compact complex Lie group G must be abelian, by considering its adjoint representation. Then invoke the last Exercise sheet to conclude that such a group must be of the form $\mathfrak{g}/\Gamma$ for a discrete group Γ.

3. Let G be a simply connected complex Lie group, let $\mathfrak{g} = \mathrm{Lie}(G)$ and let $\mathfrak{k}$ be a real form of $\mathfrak{g}$. Show that the map

$$\mathfrak{g} \to \mathfrak{g}, \quad x + iy \mapsto x - iy$$

for all $x, y \in \mathfrak{k}$ can be lifted to a real Lie group automorphism $\theta : G \to G$. If we define

$$K = G^\theta = \left\{ g \in G \middle| \theta(g) = g \right\}$$

then show that K is a real Lie group with Lie algebra $\mathfrak{k}$.

4. Find explicit Lie algebra isomorphisms:

- $\mathfrak{so}_{3,\mathbb{C}} \cong \mathfrak{sl}_{2,\mathbb{C}}$
- $\mathfrak{so}_{4,\mathbb{R}} \cong \mathfrak{so}_{3,\mathbb{R}} \oplus \mathfrak{so}_{3,\mathbb{R}}$
- $\mathfrak{sl}_{2,\mathbb{C}} \cong \mathfrak{so}_{1,3}$ (as real Lie algebras), where the Lorentz Lie algebra is

$$\mathfrak{so}_{1,3} = \left\{ X \in \mathrm{Mat}_{4\times 4}(\mathbb{R}) \middle| X^T\eta + \eta X = 0 \right\}$$

  with $\eta = \mathrm{diag}(-1, 1, 1, 1)$.